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ON THE THERMOMECHANICAL STATE OF AN UNDERGROUND SHAFT FILLED WITH A READILY MIXING LIQUID*

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Temperature fields and the thermal stresses induced by them in rock strata around a cylindrical or spherical underground shaft filled with readily mixing liquid is studied. Approximate analytical solutions are given for small dimensionless times of the quasi-static problem of thermoelasticity, which enable the shaft stability to be evaluated and enable the possibility of its disintegration to be assessed. The results of calculations of the boundary layer of the strata, and of an analysis of the dynamics of the processes in them are presented.

In view of the use of artificially created underground shafts for storing liquefied gases and other forms of liquid raw materials, the need arises to predict the heat and thermomechanical interaction of unsteady processes, due to the considerable initial temperature differences of the liquid and rock strata. The initial period is of the greatest interest, when, owing to the strong mixing of liquid after the filling up the available capacity, and intense heat convection that are the dominant factors in heat exchange in a liquid, the heat exchange in the strata is accompanied by large temperature gradients and stresses induced by them.

For the conditions considered, the effective thermal conductivity of the liquid exceeds that of the rock by several orders of magnitude, hence in the mathematical formulation of the problem we need only take into account the specific heat of the liquid. This assumption together with the assumption of perfect heat contact at the interface of the two media leads to the problem of heat conduction with boundary conditions that contain a time derivative /1/.

Let u be the unknown temperature of the rock $a \leqslant r < \infty$ referred to the initial temperature difference in the system containing a cylindrical $(\gamma = 1)$ or spherical $(\gamma = 2)$ shaft of radius a; T_m° and T_f° are the initial temperature of the rock and the liquid, λ and $\varkappa$ are the thermal conductivity and temperature diffusivity of the homogeneous elastic rock, α, E and μ are its elastic constants /2/, c and ω are the specific heat and the mass of liquid per unit surface of the shaft, and w and σ_{jj} are the radial component of the displacement vector and thermal stresses in the rock. In this notation and assumptions the problem to be solved of the thermomechanical state of an underground shaft containing readily mixing liquid can be written in the following unconnected system of equations of thermoelasticity in the quasi-static formulation:

$$-\frac{1}{r^\gamma}\frac{\partial}{\partial r}\left(r^\gamma \frac{\partial u}{\partial r}\right) + \frac{1}{\varkappa}\frac{\partial u}{\partial t} = 0 \quad (a < r < \infty,\ t > 0) \tag{1}$$

$$\frac{\partial u}{\partial t} - \sigma \frac{\partial u}{\partial r} = 0 \quad (r = a,\ t > 0) \tag{2}$$

$$u(a, 0) = 1, \quad u(r, 0) = 0, \quad \lim_{r\to\infty} u = 0; \quad \sigma = \frac{\lambda}{c\omega} \tag{3}$$

$$\frac{1}{r^\gamma}\frac{\partial}{\partial r}\left(r^\gamma \frac{\partial w}{\partial r}\right) - \frac{\gamma}{r^2} w = \alpha T_{mf}^\circ \frac{1+\mu}{1-\mu}\frac{\partial u}{\partial r} \quad (a < r < \infty) \tag{4}$$

$$\sigma_{rr}(a, t) = 0, \quad \lim_{r\to\infty} w = \lim_{r\to\infty} \sigma_{rr} = 0 \quad (\gamma = 1, 2) \tag{5}$$

Equations (4) and (5) imply that the displacements in the rock are determined in terms of the averaged temperature

$$\langle u_\gamma(r, t)\rangle = r^{-\gamma}\int_a^r x^\gamma u_\gamma(x, t)\, dx$$

by the relation

$$w_\gamma = \alpha T_{mf}^\circ \frac{1+\mu}{1-\mu}\langle u_\gamma(r, t)\rangle \quad (T_{mf}^\circ = T_f^\circ - T_m^\circ) \tag{6}$$

Then

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$$\sigma_{rr}^{(\gamma)} = -\frac{\alpha T^{\circ}_{mf} E\gamma}{(1-\mu)r}\langle u_\gamma(r,t)\rangle \tag{7}$$
$$\sigma_{\varphi\varphi}^{(\gamma)} = \frac{\alpha T^{\circ}_{mf} E}{1-\mu}\left[\frac{1}{r}\langle u_\gamma(r,t)\rangle + (1-\gamma)u_\gamma(r,t)\right]$$
$$\sigma_{\varphi\varphi}^{(2)} = \sigma_{\theta\theta}^{(2)}, \quad \sigma_{\varphi\varphi}^{(1)} = -\sigma_{rr}^{(1)}, \quad \sigma_{zz}^{(1)} = -\frac{\alpha T^{\circ}_{mf} E}{1-\mu} u_1(r,t)$$

The unconnected quasi-static problem of thermoelasticity for $\gamma = 2$ and the stationary boundary condition $u_2(a,t) = \mathrm{const}$ was considered in detail in /2/.

The unsteady problem of heat conduction (1)–(3) can be solved by the Laplace transformation, with transforms of the solutions of the form

$$U_1(r,s) = \frac{K_0(r\sqrt{s/\varkappa})}{sK_0(a\sqrt{s/\varkappa}) + \sigma\sqrt{s/\varkappa}\,K_1(a\sqrt{s/\varkappa})} \tag{8}$$

$$U_2(r,s) = \frac{a}{r}\frac{\exp[-(r-a)\sqrt{s/\varkappa}]}{s + (\sigma/\sqrt{\varkappa})\sqrt{s} + \sigma/a} \tag{9}$$

where $K_n(z)$ is the modified Bessel function. The original function of expression (8) can be theoretically obtained using the Riemann-Mellin integral by fine and coarse computations in the complex plane of the variable s, which lead to an incomplete integral of a complex combination of special functions. For the purposes noted at the beginning of the paper we can take, with sufficient accuracy for applications, in (8) instead of the function $K_n(z)$, the principal part of its asymptotic representation /3/, which leads to the formula

$$U_1(r,s) \underset{s\to\infty}{\sim} \sqrt{\frac{a}{r}}\frac{\exp[-(r-a)\sqrt{s/\varkappa}]}{\sqrt{s}(\sqrt{s} + \sigma/\sqrt{\varkappa})} \tag{10}$$

in the space of the original functions, which approximate the exact solution well when $\varkappa t/a^2 < 1$.

Transform (9) can be reduced to the following equivalent form:

$$U_2(r,s) = \left(\frac{1}{\sqrt{s}+\nu_1} - \frac{1}{\sqrt{s}+\nu_2}\right)\frac{a\exp[-(r-a)\sqrt{s/\varkappa}]}{r(\nu_2-\nu_1)} \tag{11}$$
$$\nu_{1,2} = \frac{\sigma}{2\sqrt{\varkappa}} \mp \left(\frac{\sigma^2}{4\varkappa} - \frac{\sigma}{a}\right)^{1/2} \quad (\nu_2 > \nu_1 > 0)$$

Consequently for the original functions of the required solutions we have

$$u_1 = \sqrt{\frac{a}{r}}\exp\left[\frac{\sigma}{\varkappa}(\sigma t + r - a)\right]\mathrm{erfc}\left(\frac{r-a}{2\sqrt{\varkappa t}} + \sigma\sqrt{\frac{t}{\varkappa}}\right) \tag{12}$$
$$u_2 = \frac{a(v_2 - v_1)}{r(\nu_2-\nu_1)}$$
$$v_{1,2} = \nu_{1,2}\exp\left[\nu_{1,2}\left(\frac{r-a}{\sqrt{\varkappa}} + \nu_{1,2}t\right)\right]\mathrm{erfc}\left(\frac{r-a}{2\sqrt{\varkappa t}} + \nu_{1,2}t\right)$$

By virtue of the assumptions made before, the temperature of the liquid that fills the shaft can be found from these solutions by substituting $r = a$.

Note that for a wide range of thermophysical parameters of rocks, and dimensions of shafts in (9) $\sigma/a \approx 10^{-5}$, so that for small $\varkappa t/a^2$ it can be neglected and after introducing the dimensionless quantities

$$\xi = \frac{r}{a}, \quad \tau = \frac{\varkappa t}{a^2}, \quad B = \frac{a\sigma}{\varkappa} \tag{13}$$

we can represent solutions (12) in the single form

$$u_\gamma(\xi,\tau) = \xi^{-\gamma/2}v(\xi,\tau) \tag{14}$$
$$v(\xi,\tau) = \exp[B(B\tau + \xi - 1)]\,\mathrm{erfc}\left(\frac{\xi-1}{2\sqrt{\tau}} + B\sqrt{\tau}\right)$$

We shall indicate the explicit form of functions (6) and (7), taking into account that the time t appears in these formulas as a parameters. Hence it is more convenient to carry out the calculations in the space of the transforms where the transforms have a simpler form than (12). However the direct use of (11) leads to an analytically unsolvable integral, and we must therefore start from the exact solution (8). Taking into account the value of the quadrature

$$\int_0^b x^{n+1}K_n(x)\,dx = 2^n\Gamma(n+1) - b^{n+1}K_{n+1}(b)$$

and introducing the function

$$Q_1(r,s) = \frac{aK_1(a\sqrt{s/\varkappa}) - rK_1(r\sqrt{s/\varkappa})}{sK_0(a\sqrt{s/\varkappa}) + \sigma\sqrt{s/\varkappa}K_1(a\sqrt{s/\varkappa})}$$

after using the asymptotic representation of the function $K_n(z)$ we find that for $\varkappa t/a^2 < 1$

$$Q_1(r,s) \underset{s\to\infty}{\sim} \left\{1 - \sqrt{\frac{r}{a}}\exp\left[-(r-a)\sqrt{\frac{s}{\varkappa}}\right]\right\}\frac{a\sqrt{\varkappa}}{s(\sqrt{s}+\sigma/\sqrt{\varkappa})} \tag{15}$$

In the case of $\gamma = 2$ we start the calculations directly from (9), setting $\sigma/a = 0$. As the result, we have the function

$$Q_2(r,s) = \left\{\left(a + \sqrt{\frac{\varkappa}{s}}\right) - \left(r - \sqrt{\frac{\varkappa}{s}}\right)\times\exp\left[-(r-a)\sqrt{\frac{s}{\varkappa}}\right]\right\}\frac{\sqrt{\varkappa}}{s(\sqrt{s}+\sigma/\sqrt{\varkappa})} \tag{16}$$

After the introduction of the dimensionless variables (13), and the determination of the dimensionless displacements and stresses by the formulas

$$W_\gamma = \frac{(1-\mu)w_\gamma(\xi,\tau)}{(1+\mu)a\alpha T^*_{mf}}, \quad \Sigma_{jj}^{(\gamma)} = \frac{(1-\mu)\sigma_{jj}^{(\gamma)}(\xi,\tau)}{\alpha T^*_{mf}E} \tag{17}$$

From (15) and (16) we finally obtain

$$W_1 = \frac{1}{\xi}q_1(\xi,\tau), \quad \Sigma_{\xi\xi}^{(1)} = -\frac{1}{\xi^2}q_1(\xi,\tau) \tag{18}$$

$$W_2 = \frac{1}{\xi^2}q_2(\xi,\tau), \quad \Sigma_{\xi\xi}^{(2)} = -\frac{2}{\xi^3}q_2(\xi,\tau)$$

$$\Sigma_{\varphi\varphi}^{(2)} = \frac{1}{\xi^3}q_2(\xi,\tau) - \frac{1}{\xi}v(\xi,\tau)$$

$$q_1 = \frac{1}{B}\{1 - v(1,\tau) - \sqrt{\xi}[\operatorname{erfc}(\eta) - v(\xi,\tau)]\}$$

$$q_2 = \frac{1}{B}\left\{2\sqrt{\frac{\tau}{\pi}}[1-\exp(-\eta^2)] + \left(\xi - \frac{1}{B}\right)v(\xi,\tau) + \left(1-\frac{1}{B}\right)[1 - v(1,\tau) - \operatorname{erfc}(\eta)], \quad \eta = \frac{\xi-1}{2\sqrt{\tau}}\right.$$

Formulas (14) and (18) provide a complete description of the unsteady temperature fields and of the dynamics of the quasi-static stresses they produce in the neighbourhood of an underground shaft, without taking into account the stress-strain state of the rock, due to the forces of its own weight and the presence of a closed cavity. The latter problem is studied in detail in /4,5/, and is not considered here. For an evaluation of the shaft stability and to determine the possibility of its disintegration it is necessary to carry out a simple superposition of the appropriate solutions.

From the point of view of the shaft stability the heat interaction is of a local character /4/, and hence it is of the greatest practical interest to determine the stress-strain state of the layer immediately adjacent to it ($\xi - 1 \ll 1$). Some of the results are tabulated below where we show the temperature of the rock in the neighbourhood of the cylindrical (in italics) and spherical shafts multiplied by a factor of 10^4, and also in Figs.1–3 (the continuous lines relate to $\gamma = 1$ and the dashed ones to $\gamma = 2$, and curves *1–5* correspond to $\tau = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1$). A qualitiative and quantitative analysis of the dynamics of the process, carried out on the basis of these results enables us to draw the following conclusions.

Table 1

τ	$\xi - 1 = 0.01$		0.05		0.10		0.50	
10^{-3}	*7885*	7846	*2504*	2444	*237*	226	*0*	0
10^{-2}	*8285*	8244	*6323*	6171	*4154*	3961	*3*	2
10^{-1}	*6724*	6691	*6204*	6054	*5568*	5309	*1687*	1377
1	*3782*	3764	*3665*	3604	*3523*	3359	*2547*	2079

1) For fixed $\xi = \xi^\circ > 1$ the rock temperature monotonically approaches in time the temperature of the liquid, but without attaining it, reaching the maximum value for the given ξ° at the stationary point $\tau = \tau^\circ$. When $\tau > \tau^\circ$ the natural temperature field of the rock is restored, and the temperatures of the rock and liquid asymptotically approach zero. The temperature maxima lie on a monotonically decreasing curve which in (ξ, τ) variables can be constructed by solving the transcendental equation

$$B^2 \operatorname{erfc}(\eta + B\sqrt{\tau}) = \left(B - \frac{\eta}{\sqrt{\tau}}\right)\frac{\exp[-(\eta+B\sqrt{\tau})^2]}{\sqrt{\pi\tau}} \qquad (19)$$

The quasi-steady mode of heat conduction in the rock layer $0 \leqslant \xi - 1 \leqslant 0.1$ adjoining the shaft surface occurs when $\tau \approx 10^{-2}$, and here the temperature curves for $\gamma = 1$ and $\gamma = 2$ differ insignificantly (see Table 1). In the calculations we always used the value $B = 1.1875$ obtained from actual heat-exchange conditions.

2) In the formulation considered here the shaft contour, independently of the temperature field, is not deformed (Fig.1). When $\xi > 1$, the displacements in the rock have a maximum at the point (ξ^*, τ^*) which does not coincide with the extremum of the function $u_\gamma(\xi, \tau)$. It can be determined by solving the transcendental equations

$$\sqrt{\xi}\left\{\exp[B(B\tau+\xi-1)]\left[\left(B-\frac{\eta}{\sqrt{\tau}}\right)\frac{\exp[-(\eta+B\sqrt{\tau})^2]}{\sqrt{\pi\tau}} - B^2\operatorname{erfc}(\eta+B\sqrt{\tau})\right] + \frac{\eta\exp(-\eta^2)}{\tau\sqrt{\pi}}\right\} = B\left[\frac{1}{\sqrt{\pi\tau}} - Bv(1,\tau)\right] \text{ for } \gamma=1 \qquad (20)$$

$$\left(\xi-\frac{1}{B}\right)\exp[B(B\tau+\xi-1)]\left\{B\operatorname{erfc}(\eta+B\sqrt{\tau}) - \left(B-\frac{\eta}{\sqrt{\tau}}\right)\frac{\exp[-(\eta+B\sqrt{\tau})^2]}{\sqrt{\pi\tau}}\right\} - \frac{(1+\eta^2\tau)\exp(-\eta^2)}{\sqrt{\pi\tau}} = \frac{1}{\sqrt{\pi\tau}}\left\{\left(1-\frac{1}{B}\right)\left[B^2 v(1,\tau) + \frac{\eta\exp(-\eta^2)}{\sqrt{\tau}} - B\right] - 1\right\} \text{ for } \gamma=2 \qquad (21)$$

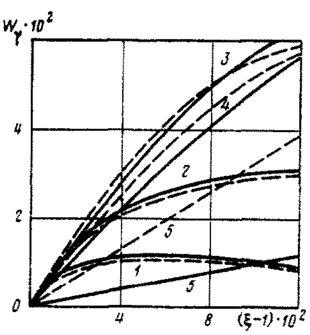

Fig.1

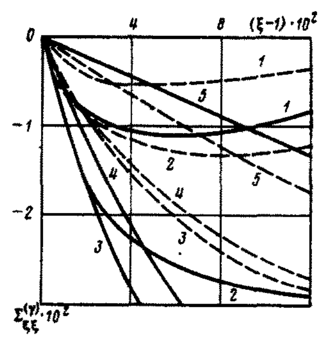

Fig.2

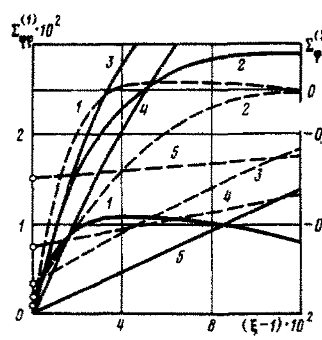

Fig.3

The difference between the numerical values of the displacements for $\gamma = 1$ and $\gamma = 2$ in the layer $0 \leqslant \xi - 1 < 0.1$ essentially depends on time (it increases as τ increases).

3) When the liquid exerts no pressure in the shaft its surface is free of normal temperature stresses (Fig.2) for all $\tau > 0$. When $\xi > 1$ a considerable difference is observed in the magnitude of compressive stresses for $\gamma = 1$ and $\gamma = 2$ whose dynamics include reaching an extremal value at the fixed point ξ. The instant of time τ^* that corresponds to the given point ξ^* of the rock is calculated using Eqs.(20 and (21).

4) The tangential stresses (Fig.3) at $\xi \geqslant 1$ differ substantially for $\gamma = 1$ and $\gamma = 2$: $\Sigma^{(1)}_{\varphi\varphi}(1, \tau) = 0$, and the $\Sigma^{(2)}_{\varphi\varphi}$ is proportional to the temperature of the shaft surface. When $\xi \geqslant 1$ these stresses differ in magnitude and sign: whereas when $\gamma = 1$ for all ξ and τ only stretching tangential stresses are observed, for $\gamma = 2$ they increase monotonically as ξ increases in the region of negative values, reach a positive maximum, and then approach steady negative values. Numerical experiments showed that the magnitude of the positive maximum of $\Sigma^{(2)}_{\varphi\varphi}$ is of the order of 10^{-3}.

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AXISYMMETRIC CONTACT PROBLEMS FOR NON-UNIFORMLY AGING LAYERED VISCOELASTIC FOUNDATIONS*

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Contact problems for non-uniformly aging multilayered viscoelastic foundations are studied. It is assumed that the thickness of the top layer is much less than the characteristic dimension of the area of contact. Integral equations of mixed problems containing Fredholm and Volterra operators are derived and a method for solving them is given. Basic versions of the non-uniform aging of a packet of layers are studied and the case in question is compared analytically with the classical case. Numerical computations of the characteristic parameters are given.

1. We consider the contact problems of the frictionless impression of a rigid circular stamp, using a constant force P, into a multilayered non-uniformly aging viscoelastic foundation consisting of:

1) a thin non-uniformly aging layer and a uniformly aging layer of arbitrary thickness, without friction between the layers;

2) a thin layer, a non-uniformly aging core foundation and a uniformly aging layer, with the first two layers coupled to each other, and resting on the third;

3) the packet is composed of the layers listed in 2), with zero friction between them.

We shall call the layer thin if the characteristic dimension of the part of the layer subjected to the active load is much greater than its thickness. The layer thicknesses are h, l and H respectively. Smooth contact or coupling with the non-deformable support occurs at the lower edge of the multilayer packet. The surface of the stamp base is described by the function $g(r)$, and the region of contact by the inequality $r \leqslant a$.

We write the equations of state of the layer materials in the form /1/

$$e_{ij}(t, r, z) = (1+\nu)\left[\frac{S_{ij}(t, r, z)}{E} - \int_{\tau_0}^{t} \frac{S_{ij}(\tau, r, z)}{E} K(t+\varkappa(z), \tau+\varkappa(z))\, d\tau\right]$$

$$\varepsilon(t, r, z) = (1-2\nu)\left[\frac{\sigma(t, r, z)}{E} - \int_{\tau_0}^{t} \frac{\sigma(\tau, r, z)}{E} K(t+\varkappa(z), \tau+\varkappa(z))\, d\tau\right]$$

$$K(t, \tau) = E\frac{\partial}{\partial\tau} C(t, \tau)$$

Here $e_{ij}(t, r, z)$ and $S_{ij}(t, r, z)$ are the deviators of the strain and stress tensors, respectively, $3\varepsilon(t, r, z)$ is the volume strain, $\sigma(t, r, z)$ is the mean hydrostatic pressure, $K(t, \tau)$ is the tensile creep kernel, $C(t, \tau)$ is a measure of the creep, r and z are cylindrical coordinates of a point of the body, τ_0 is the time of application of the load, $\varkappa(z)$ is the non-uniform aging function, and E and ν denote the instantaneous modulus of elasticity and

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